Theoretical Modelling of Early Black Holes

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Massive black hole growth basics

The growth of black holes

gas accretion



AGN/quasars

stellar accretion



Tidal Disruption Events

MBH-MBH mergers



Gravitational waves

The growth of black holes



Mergers: just reshuffle the distribution of masses => total mass density in MBHs is constant in time

Accretion: adds external matter => total mass density in MBHs grows with time

Soltan's argument: mass density increases by > one order of magnitude in the last ~10 Gyr: accretion leads (Yu & Tremaine 2002)

What's going on at high redshift?!?!?!



Soltan's argument

$$L = \epsilon \dot{M}_{in}c^{2}$$
A fraction ϵ of mass goes into
radiation
$$\dot{M}_{acc} = (1 - \epsilon)\dot{M}_{in}$$
Only a fraction $(1 - \epsilon)$ goes into
the BH
$$L = \frac{\epsilon}{1 - \epsilon}\dot{M}_{acc}c^{2}$$
Luminosity=energy per unit
time
$$E = \frac{\epsilon}{1 - \epsilon}M_{acc}c^{2}$$

Soltan's argument

$$\Phi(L) = \frac{dN}{dLdV}$$

Luminosity function of quasars/AGN

$$\int_{t}^{0} dt \int_{0}^{\infty} \Phi(L, t) L dL = u_{tot,QSO}(t)$$

$$E = \frac{\epsilon}{1 - \epsilon} M_{acc} c^2$$

Total energy density emitted by accreting MBHs

Luminosity=energy per unit time

$$E_{tot,QSO}/V \equiv u_{tot,QSO} = \frac{\epsilon}{1-\epsilon} (M_{acc,tot,QSO}/V)c^2 \equiv \frac{\epsilon}{1-\epsilon} c^2 \rho_{acc,tot,QSO}$$

$$ho_{acc,tot,QSO}(z) = rac{1-\epsilon}{\epsilon c^2} \int_z^\infty \left|rac{dt}{dz}\right| dz \int_0^\infty \Phi(L,z) L dL$$

The growth of black holes



Low and high mass MBHs have statistically different growth histories in terms of mergers vs accretion (Dubois, Volonteri & Silk 2014, Kulier et al. 2015)

Accretion leads the overall mass density (Yu & Tremaine 2002)

Fraction of mass gained through MBH-MBH mergers: $f_{merge} = \Delta M_{merge} / M_{BH}$

The Eddington limit

The outward radiation pressure equals to the inward gravitational force

$$a_{rad}(r) = \frac{\kappa(\rho, T)L(r)}{4\pi r^2 c} \qquad \qquad g(r, t) = -\frac{GM_{\bullet}(t)}{r^2}$$

If $a_{rad} > |g|$ radiation pushes away the gas, and further accretion is halted

In reality no spherical symmetry: accretion discs



Note: usually opacity calculated via Thomson cross section. Interesting things happen with dust (see Fabian+08 for spherical symmetry and Venanzi+20 for discs; see Ferrara+24 for an application to galaxies – in spherical symmetry – and Volonteri+24 for an application to AGN/LRDs – also in spherical symmetry)

Highly super-Eddington accretion does not imply highly super-Eddington luminosities

Trapping of radiation: the time for photons to escape the disk exceeds the timescale for accretion

Luminosity suppressed (low radiative efficiency ϵ)



Caveats: - Unclear how small ϵ can really be (Jiang+19)

- SuperEddington can create powerful jets that push gas away — the duty cycle can be low (Regan+19; Massonneau+23) but with sufficiently deep potential wells feedback is less damaging (Lupi+24; Husko+24) With superEddington you get more bang for your buck: short phases of superEddington accretion can grow the MBH more than prolonged phases of Eddington-limited accretion (Massonneau+23, Husko+24)



Caveat:

Mass loss within the accretion disc can limit the mass accreted onto the MBH to a small fraction of the accretion rate at the edge of the disc. In the ADIOS model (Begelman11) the loss is such that the MBH growth rate never exceeds the Eddington rate! See Volonteri+15 and Hu+22 for analytical models accounting for mass loss

Feedback

Stars form in gas clouds where gas is very dense and cold

Stars emit ultraviolet light and then explode as supernovae

The energy "given back" heats and rarefies gas preventing further star formation and black hole growth



Feedback

The energy "given back" when a massive black hole accretes gas also heats and rarefies the gas preventing further star formation and black hole growth



Black Hole vs Galaxy Mass: feedback?

"The binding energy of the galaxy is $E_{gal}\approx M_{gal}\,\sigma^2$

The energy released by the growth of the black hole is $E_{BH}\approx 0.~1M_{BH}c^2$

The mass of the MBH is $\sim M_{BH} \sim 10^{-3} M_{gal}$

 $E_{BH}/E_{gal} \sim 10^{-4} (c/\sigma)^2 > 80$

If even a small fraction of the energy can be transferred to the gas, then an AGN can have a profound effect on the evolution of its host galaxy"

Adapted from Fabian (2012)



Slide credit: Rebekka Bieri

The feeding/feedback cycle

Feeding: the galaxy feeds the black hole through gas inflows

<u>Feedback</u>: the kinetic and radiative output from active black holes

A black hole grows by accreting gas and becomes active

An active black hole launches winds or jets that interact with the gas of the galaxy, modulating gas accretion onto the black hole and the formation of stars in the galaxy

If gas is prevented from going back to the black hole, the active black hole goes back to quiescence

Gas cools and can form stars and feed the black hole: the cycle restarts

Modeling MBH evolution

Analytical models

- clean, elegant, easily reproducible
- PS-style approach, or specific problems

Semi-analytical models

- fast, cover large parameter space, give a good physical intuition of the general astrophysics
- lack spatial information, can only use simplified analytical functions

Simulations

- naturally include spatial information and can reach a high-level of complexity
- high computational costs

(Semi)analytical models

- The advantage of an analytical approach is that in principle it has unlimited spatial resolution
- One looses control on non-analytical processes (those that cannot be described by well behaved mathematical functions, e.g., galaxy mergers)

(Semi)analytical models



Empirical/HOD models

Populate DM halos with galaxies and MBHs

Ensemble population information compared to (many!) observables to find how population evolves



Cosmological simulations

Simulations are like observational surveys: you can have either

large and shallow (large volume/many objects/low resolution/massive galaxies)

or

small and deep (small volume/few objects/high resolution/dwarf galaxies)

Classic: uniform volume

- Simulate a given volume of the Universe with the same resolution everywhere (still with local refinement)
- Need for large volume (to probe the large scale structure) limits the resolution
- Normally spatial resolution ~1kpc, particle mass resolution ~10⁶ M_{sun} (and one needs at least 50 particles for a galaxy)
- Running a cosmological simulation with these specs takes several tens of millions of CPU hours
- Several months to years of real time

Elegant zooms

- A specific area in a uniform cosmological simulation is resampled and re-simulated at higher mass and force resolution.
- Improve in the global cost of the project, higher resolution for the same number of particles
- Since only a small volume is resimulated, the number of halos in the volume is small => loose statistics





Trebitsch+21



Trebitsch+21

Simulations vs (semi)analytical

- what type of technique best to use to study a given physical process
- how you approach the problem
- physical intuition

The goals of theoretical models of massive black holes



Credit: Michael Tremmel



What I've learnt from simulations that I had not learnt from semianalytical models

Things that are not easily described by analytical formulae

Things related to "messy" conditions or driven by local effects or driven by environment

E.g., the importance of the central galaxy density/nucleus on the pairing fraction of MBHs, messiness of high-z galaxies making dynamical friction timescales useless, SNe messing up gas near MBHs, cosmological tidal field influencing MBH growth

References for semi-analytical models: Kauffman & Haehnelt 00; Volonteri+03; Croton+96; Somerville+08; Fanidakis+11; Dayal+19; Fontanot+20; Izquierdo-Villalba+20; Trinca+22 etc etc <u>Generally all processes have been first implemented in SAMs, then in simulations</u>

1. Mimic the formation of black holes

1.1 Where?

In high gas and stellardensity regions, mimicking MBH formation mechanisms (e.g., Tremmel+17; Habouzit+17) Or put BHs in halos above a given threshold using an on-the-fly halo finder (e.g., Di Matteo+08)

1.2 With what initial seed BH mass ?

Models predict ~ 10^3 - 10^5 Msun; normally choice limited by mass resolution => 10^5 - 10^6 Msun

See Habouzit+20, 22 for a comprehensive analysis of the differences between subgrid models

2. Mimic gas accretion onto MBHs

Bondi-Hoyle-Littleton accretion rate (e.g., Di Matteo+08) Or torque limited (e.g., Angles-Alcazar+17)

$$\dot{M}_{\rm BH} = \alpha 4\pi G \rho \frac{M_{\rm BH}^2}{(c_{\rm s}^2 + v_{\rm rel}^2)^{3/2}}$$

Fudge factor Gas density Sound speed required at low

required at low resolution to capture high accretion rates due to unresolved large density contrasts

See Habouzit+20, 22 for a comprehensive analysis of the differences between subgrid models

Or flux accretion

(only at very very very resolution, e.g., Regan+19; Angles-Alcazar+21)

$$\dot{M}_{\rm accr} = -\int_{\Delta V} \nabla \cdot (\rho_{\rm gas} \mathbf{v}_{\rm rel}).$$

2. Mimic gas accretion onto MBHs

Capped at the Eddington luminosity (e.g., Di Matteo+08)

 $\dot{m}_{\rm Edd} = rac{4\pi G m_{\rm BH} m_{\rm p}}{\epsilon_{\rm r} \sigma_{\rm T} c}$

the outward radiation pressure equals the inward gravitational force $a_{rad}(r) = \frac{\kappa(\rho, T)L(r)}{4\pi r^2 c} \quad g(r, t) = -\frac{GM_{\bullet}(t)}{r^2}$

See Habouzit+20, 22 for a comprehensive analysis of the differences between subgrid models

Or not, and include superEddington feedback (e.g., Regan+19; Massonneau+22; Huska+24)

3. Mimic dynamics of MBHs

Advect MBHs to the minimum of the local potential (e.g., Di Matteo+08) Or account for the "missing" force – dynamical friction (e.g., Dubois+13; Tremmel+15; Pfister+19)

$$\vec{F}_{\star} = -4\pi \mathbf{G}^2 M_{\bullet}^2 \frac{\vec{v_{\bullet}}}{v_{\bullet}^3} \left\{ \ln \Lambda \int_0^{v_{\bullet}} 4\pi v^2 f(v) dv + \dots \int_{v_{\bullet}}^{\infty} 4\pi v^2 f(v) \left[\ln \left(\frac{v + v_{\bullet}}{v - v_{\bullet}} \right) - 2 \frac{v_{\bullet}}{v} \right] dv \right\}$$

with:

$$\ln \Lambda = \ln(4\Delta x/r_{\text{def}})$$

$$4\pi v^2 f(v) = \frac{3}{256\pi\Delta x^3} \sum_{i \in S} m_i \delta(v_i - v_i)$$



4. Mimic MBH mergers

Proximity criterion: two MBHs must be closer than N resolution elements (N ⇔ "kernel") (e.g., Di Matteo+08) => hundreds of pc to kpc, 3-5 orders of magnitude larger than separation at which MBHs actually merge

Can also add a dynamical criterion, e.g., the MBHs are bound (e.g., Tremmel+17)

Caveat: key orbital parameters such as mass ratio and eccentricity near merger, necessary for waveform modeling, data analysis and electromagnetic signatures cannot be fully predicted

Dynamics and MBH mergers

KETJU (e.g., Mannerkoski+19,20,21,23; https://www.mv.helsinki.fi/home/phjohans/ group-website/research/ketju/)

- SMBHs are resolved as point particles without gravitational softening
- Dynamical friction and hardening of SMBH binaries from interactions with stellar particles are directly captured
- Post-Newtonian dynamics of SMBH binaries, such as orbital decay from GW emission and precession of the orbit
- The main idea in KETJU is to add small spherical regions (dashed circles) with typical radii of ~ 10 pc centred on the SMBHs (shown as black dots in the two insets), where the dynamics are integrated using a high-accuracy regularised integrator

RAMCOAL (Li,Volonteri+24)

- Sub-grid treatment of MBH binary dynamics and accretion
- Uses local quantities to calculate local potential
- Sub-grid model of stellar density makes it almost resolution-independent out to 100 pc resolution. Massive Black Holes merge at 10-3 pc: gain of 5 orders of magnitude!
- includes dynamical friction, stellar hardening, migration in circumbinary disc, GW emission, separate accretion on each MBH in the binary

5. Mimic AGN feedback

$$L_{\rm AGN} = \epsilon_f \epsilon_r \dot{M}_{\rm BH} c^2$$

Free parameter

Mass-radiative energy conversion (depends on spin)

Thermal input: increase the gas temperature by distributing specific energy in a small sphere near the MBH (e.g., Di Matteo+08) Kinetic input: inject outflows with high velocity (typically 1e4 km/s) close to the MBH. Gas is heated and ejected (e.g., Choi+12) Radiative input: radiation transfers momentum to particles/cells. Generally effective in the presence of dust (e.g., Novak+12; Bieri+17; Costa+18)

See Habouzit+20, 22 for a comprehensive analysis of the differences between subgrid models
AGN in cosmological simulations

AGN feedback is <u>very good</u> at suppressing MBH growth In simulations, feedback is stronger at higher resolution \rightarrow one needs to decrease the efficiency as resolution increases



Energy injection is more effective at higher resolution, because the mass to be heated and/and swept up is smaller

$$\dot{M}_{\rm accr} = -\int_{\Delta V} \nabla \cdot (\rho_{\rm gas} \mathbf{v}_{\rm rel}).$$

Lupi+19, see also Negri & Volonteri 17; Biernacki+17

AGN in cosmological simulations

6. Evolve MBH spins (e.g., Dubois+14)

By MBH-MBH mergers, using results from GR simulations. A series of consecutive mergers tends to give spin ~0.7

By gas accretion considering the relative direction of spin and gas angular momentum. Angular momentum is transferred from disk to hole, spin up or down depends on disk/hole mass ratio and alignment as well as on spin extraction in jets





Modeling MBH evolution

Key takeways/points for discussion:

- One needs to choose the right technique for a given problem
- Generally relative good agreement between models: see Habouzit+20, 22 for a comprehensive analysis of the differences between subgrid models
- But alas we're far from having a complete model
- Resolution is a big problem

Massive black hole growth

MBH growth: AGN feedback

AGN feedback is known to regulate star formation in massive galaxies AGN feedback is also <u>very good</u> at suppressing its own MBH growth

- Density:
 - Strong decrease with jets
 - $\circ\,$ Mild decrease with winds
- Immediate temperature increase **L** \dot{M}_{acc} **L** Reminder: $\dot{M}_{BHL} = \frac{4\pi G^2 M_{BH}^2 \bar{\rho}}{(\bar{c}_8^2 + \bar{v}_{rol}^2)^{3/2}}$
- Sharp drops down to very sub-Eddington regimes



Slide credit: W. Massonneau

MBH growth: SN feedback

In dwarf galaxies, supernova feedback is also able to suppress star formation and MBH accretion (Dubois+15; Habouzit+17; Bower+17; Angles-Alcazar+17; Prieto+17; McAlpine+17,18 etc)



Ramses cosmological zoom, Mh=6e10 Msun @ z=6, ~5pc resolution, Prieto et al. 2017



SNe cause rapid, dramatic fluctuations in the gas density near the MBH



Costa et al. 2014, Prieto et al. 2017

SN feedback and MBH growth: nuclear star clusters at the rescue (?)



3. The most massive NSCs are the densest known stellar systems, and can reach mass surface densities of $\sim 10^6~M_\odot/{
m pc}^2$ or more. This is true even for reliable (i.e., spectroscopically derived) NSC masses, which suggests that the $\sim 10^5~M_\odot/{
m pc}^2$ upper limit in surface density suggested by Hopkins et al. (2010) to be due to stellar feedback may need to be revised upward. These dense massive clusters have masses derived primarily from the stellar population fits of Rossa et al. (2006).

Low-mass BHs grow more when in a nuclear star cluster



Neumayer+20

MBH growth in dwarfs

Dwarf galaxies don't grow much themselves

Their MBHs also cannot grow much (statistically)

$$M(t) = M_{in} e^{(\frac{1-\varepsilon}{\varepsilon}f_{\rm Edd}\frac{t}{0.45\rm Gyr})}$$

If we observe MBH~10⁵ Msun today, its average accretion rate must have been highly subEddington (or the duty cycle very low)

MBH growth in dwarfs

Dwarf galaxies don't grow much themselves

Their MBHs also cannot grow much (statistically)



Massive black hole growth



MV+21

The growth of black holes

gas accretion



AGN/quasars

stellar accretion



Tidal Disruption Events

MBH-MBH mergers



Gravitational waves

Merging massive black holes and gravitational waves



Merging massive black holes and gravitational waves

MBHs grow along with galaxies through accretion and MBH-MBH mergers

LISA can see black holes at very very early cosmic times: when they form



LISA for ESA Cosmic Vision, Amaro Seoane+13

The journey of two black holes



Slide credit: Hugo Pfister

MBH dynamics and MBH mergers



Slide concept: Monica Colpi

The erratic dynamical life of MBHs in dwarf galaxies



In a smooth potential, e.g., isothermal sphere, dynamical friction causes orbital decay towards the center (Binney & Tremaine) ON :

$$t_{\rm df} = 0.67 \,\rm{Gyr} \left(\frac{a}{4 \,\rm{kpc}}\right)^2 \left(\frac{\sigma}{100 \,\rm{km \, s^{-1}}}\right) \left(\frac{M_{\rm BH}}{10^8 \,M_{\odot}}\right)^{-1} \frac{1}{\Lambda}$$

Many dwarf galaxies – especially at high z – have messy, non smooth, time-variable potentials, and no real center (Pfister+19,Ma+21): erratic dynamics





Erratic dynamics: nuclear star clusters at the rescue



Stars stripped from one nucleus provide additional dynamical friction and speed up subsequent hardening ("Ouroboros Effect", Ogiya+20)

Wandering black holes



A natural corollary of difficult binding and erratic dynamics is the existence of wandering MRHs



Governato+94; Schneider+02; Volonteri+03, 05; Bellovary+10; Tremmel+18; Di Matteo+23

The growth of black holes

gas accretion



AGN/quasars

stellar accretion



Tidal Disruption Events

MBH-MBH mergers



Gravitational waves

Tidal disruption events and stellar accretion



$$a_{self} < \Delta a_t$$

$$<=>\frac{GM_{\star}}{R_{\star}^2} < \frac{GM_{\bullet}}{r^2} \frac{R_{\star}}{r}$$

$$<=>r < R_{TDE} = R_{\star} \left(\frac{M_{\bullet}}{M_{\star}}\right)^{1/3}$$

$$<=>r < 6 \times 10^5 \text{km} \left(\frac{M_{\bullet}}{M_{\odot}}\right)^{1/3}$$

$$\begin{split} R_{TDE} &> R_{Sch} \\ \Leftrightarrow 6 \times 10^5 \mathrm{km} \left(\frac{M_{\bullet}}{M_{\odot}}\right)^{1/3} &> 3 \mathrm{km} \left(\frac{M_{\bullet}}{M_{\odot}}\right) \\ \Leftrightarrow 10^8 &> \frac{M_{\bullet}}{M_{\odot}} \\ \end{split}$$
Courtesy of Hugo Pfister

Tidal disruption events and stellar accretion

From the galaxy stellar density profile + MBH mass one can derive expected rates (Wang & Merritt 04, Stone & Metzger 16)



Tidal disruption events in dwarf galaxies

Sample of local galaxies with dynamically measured MBH masses and welldetermined central stellar distribution

Down to low MBH/galaxy mass (Nguyen+18)

Some with a resolved Nuclear Star Cluster (including MW, Schodel+18)



Caveat: deprojected Sersic profiles are rather shallow — lower limit to TDE rates

See Stone & Metzger 16 for a comprehensive model for more massive galaxies

Tidal disruption events in local galaxies



Significant scatter — a variety of galaxies and MBHs

Presence of a nuclear star cluster increases TDE rates by ~2 orders of magnitude

TDE rate ~10⁻⁴-10⁻⁵ per galaxy per year, assuming that all galaxies host a MBH (see Stone & Metzger 16 for effect of MBH occupation fraction)

Dropping at M_{gal} > 10¹¹ M_{sun} because M_{BH} can be > 10⁸ M_{sun}



Pfister+20

Tidal disruptions and MBH growth

Tidal disruptions can contribute significantly to MBH growth up to ~ 10^{5} - 10^{6} Msun (Alexander & Bar Or 17)

Provided that galaxies have dense central stellar densities and MBHs stay put in the galaxy center



MBH growth basics

Key takeways/points for discussion:

- Gas accretion is expected to be the main growth channel, except in specific cases (basically when gas accretion is inefficient)
- The Eddington limit is an actual limit only in spherical symmetry but beware of feedback
- AGN feedback regulates both star formation and MBH growth
- SNe limit MBH growth in low-mass galaxies (?)
- Do MBHs merge efficiently in high-z galaxies?
- Stellar accretion is generally subdominant

Massive black hole growth at high redshift

Massive Black Hole seeds

We can consider these processes as a continuum in physical terms



The rare heaviest seeds formed in strong UV radiation sites sit in the center of an almost spherical gas distribution and have the highest masses at birth



More common rapid halo growth with mild UV radiation leads to a more asymmetric mass distribution and less extreme stellar masses



The very common PopIII remnants are the lightest at birth and are scattered in a morphologically complex gas distribution and shallow potential well

The growth of seeds



The rare heaviest seeds initially sit in the center of an almost spherical gas distribution, but in satellites



The initial growth of heavyish seeds is challenged by their turbulent environment



The very common lightest seeds are scattered in a morphologically complex halo with few pockets of dense gas

Regan& MV 2024; see also Smith+18, Pfister+19, Ma+20, Chon+16, 18, 21; Shi+24, Mehta+24

The growth of seeds: rare heaviest seeds





High LW radiation produced in nearby halo => MBH is in a satellite

Chon+18

- (i) radiative feedback from the MBH itself suppresses the accretion rate (no kinetic feedback is included)
- (ii) the intense supernova activity injects a large amount of energy into the gas, and causes the supersonic turbulence
- (iii) the BH accelerates when falling into the galactic potential well, and obtains a large velocity relative to the gas.
- (iv) the BH is still wandering within the galactic potential at the end of the simulation.



Chon+16, 18, 21

The growth of seeds: rare heaviest seeds



Latif+20

More optimistic results are found if the MBH is placed (directly) in the center of a central halo – even in the presence of feedback from SNe. AGN feedback is here included only as X-ray radiation, so it's a weak feedback.

The growth of seeds: lighter seeds

The large majority of light seeds is unable to grow (Smith+18) because of:

- random motions in shallow and irregular potential wells with turbulent gas
- stellar feedback
- BH feedback

Only about 0.1-1% is able to grow to >1e4 Msun when

- BH feedback is not very efficient (Shi+23,24)
- gas densities are so high that potential well deepens before SNe explode (Shi+23,24) or

- SNe push gas towards the BHs (Mehta+24)





The growth of seeds: beyond gas accretion



Cosmological simulation of a $10^{10} M_{\odot}$ galaxy at z = 6

State of the art subgrid physics: cooling, star formation, SNae, metal enrichment, etc...

Spatial resolution of $\Delta x=7pc$ MBHs can accrete and exert feedback

Dynamical friction implemented explicitly: MBH respond to a local, evolving gravitational potential

Pfister+20

Tidal disruptions and MBH growth



Stellar accretion can be important early on when gas accretion is hampered by SNe

Stellar accretion is less variable than gas accretion: rate $\sim 10^{-5}/yr$

Off-center MBHs can also be a source of TDEs — especially following galaxy mergers

Pfister+20b

Tidal disruptions and MBH growth

The combination of accretion of stars and mergers with stellar BHs, plus gas accretion are key to help MBHs grow fast (Kritos+24,25)

Some Monte-Carlo codes now find that stellar accretion alone can grow MBHs to >1e7 Msun (!!!) (Zhang & Amaro-Seoane 25)

 \rightarrow this is in clusters with mass 1e9 Msun and half-mass radius <10 pc!

Tidal disruptions and LRDs

If stellar densities are very high (Baggen+24; Akins+24) can expect high stellar accretion rates

But be careful about how to calculate rates! Need to model the MBH and cluster self-consistently

Estimates of MBHs from runaway collapse >> LRD number densities With a TDE rate of ~1e-4/yr possible to explain number of LRDs (Bellovary+25)

Caveats:

- reaching luminosity >1e44 erg/s is difficult
- light curve: luminosity should decrease in ~ 1x(1+z) yr
MBH growth at high-z

Key takeways/points for discussion:

- Growing seeds is really really really hard!!!
- Stellar accretion can help, but not sure yet if as much as desired